Determinants

A matrix is invertible when determinant is zero
and is singular when determinant is non-zero.

Properties of Determinants

① $\det I = 1$

② Reverse sign of determinant when we exchange rows

③a $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ Row-Wise linearity

③b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ Row-Wise Linearity

④ If any 2 rows in a matrix are equal then the det. is zero
(To see this, apply #2, exchange the equal rows and we have the same matrix but we must reverse the sign. Only true for zero!)

⑤ The determinant does not change when subtracting a multiple of row i from row j

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} \underset{3b}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -la & -lb \end{vmatrix} \underset{3a}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l \begin{vmatrix} a & b \\ a & b \end{vmatrix} \underset{4}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

⑥ If a matrix has a row of all zeros, the determinant is zero

⑦ Given an upper triangular matrix, the determinant is the product of the pivots (diagonal entries)

$$\det(U) = \begin{vmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{vmatrix} = d_1 d_2 d_3 d_4$$

To compute the determinant of a matrix, Matlab will do elimination until in upper-triangular then perform product of pivots.

⑧ The determinant of a matrix A is zero when A is <u>singular</u>.

That is, the determinant of A, $\det(A) \neq 0$ when A is <u>invertible</u>.

In the singular case we have a row of zeros, see property #6
In the invertible case we go to upper-triangular then to diagonal matrix, property #7

Recall elimination, here on a 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{elimination}]{} \begin{bmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{bmatrix}$$

the determinant of this matrix is $ad - bc$

⑨ $\det(AB) = \det(A)\cdot\det(B)$

in application we see... $\det(A^2) = [\det(A)]^2$

$\det(2A) = 2^{\#\text{rows}}[\det(A)]$ by property #3a

⑩ $\det(A^T) = \det(A)$

To see this, $|A^T| = |A|$

$|U^T L^T| = |LU|$

$|U^T||L^T| = |L||U|$