Linear Independence, Spanning a space, basis, dimension

<u>FACT</u>: If A is an $m \times n$ matrix w/ $\underline{m < n}$
(that is, more unknowns than equations),
then there are nonzero solutions to $Ax = 0$.

This is because there will be free variables!

INDEPENDENCE

Vectors $x_1, x_2, \ldots, x_n$ are <u>linearly independent</u> if:

- no combination gives zero vector (except trivial coeff's of 0's)

$$c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \neq 0$$

that is, when $v_1, \ldots, v_n$ are columns of A, they are <u>independent</u> if nullspace of A is zero vector.
They are <u>dependent</u> if $Ac = 0$ for $c \neq 0$.

- Independent if rank $= n$ (# of rows) ... no free var's
- Dependent if rank $< n$ (# pivots < # rows)

SPANNING A SPACE

vectors $v_1, \ldots, v_n$ <u>span</u> a space means:

- the space consists of all comb's of vectors v

* A <u>basis</u> for a vector space is a sequence of vectors $v_1, v_2, \ldots, v_d$ with 2 properties

1. Vectors are independent
2. vectors span the space

<u>Example</u>: space is $\mathbb{R}^3$

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

1. are these <u>independent</u>?

$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

All c's must be zero. So, <u>YES</u>!

1. Or, check <u>nullspace</u>:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Only vector in null space is zero vector, so <u>independent</u>!

Another basis is: <u>NOT!</u>

$$\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$$

rows 1, 2 not independent, So not invertible! So cols not independent

WHEN DO WE HAVE A BASIS?

* In $\mathbb{R}^n$, n vectors give a <u>basis</u> if the $n \times n$ matrix formed from col's of vectors is <u>invertible</u>.

Every basis of a space has the same number of vectors (that number is the DIMENSION).

Example:

Space is C(A) column-space

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

N(A)

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

- Our matrix spans the column space
- BUT the null space is not empty so not independent

- there are 2 independent columns so rank of $A = 2$

* Rank(A) = # pivot col's = dimension of c(A)

- A basis of the column-space is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

- the dimension of the column space is the rank

$$\mathrm{DIM}(C(A)) = R$$

- the dimension of the null space is the # of free variables

$$\mathrm{DIM}(N(A)) = n - R$$