Algorithm for finding null space $Ax = 0$

Example : [ELIMINATION]

pivot

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Note: col 1, col 2 not independent
row 1 + row 2 = row 3
all this will come out of elim.

* the goal is to get 0's in col 1 excluding the first row

$r_2 - 2r_1$, $r_3 - 3r_1$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

1. check here for non-zero "pivot"
2. if non-zero do a row exchange

* from 1,2 we know col 2 is dependent on col 1

3. New pivot in col 3

$r_3 - r_2$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

a row of zeros means it is lin. dependent

pivot cols — free cols

U is in echelon form
U has 2 pivots (rank = 2)
U has 2 free columns

Rank of A = # of pivots

- columns 2 and 4 are free - I can assign any value to them in the solution, then I solve for the pivots

Solutions:

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$U \quad \begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_3 + 4x_4 &= 0 \end{aligned}$$

$x_2 = 1$ $\quad x_4 = 0$

Here I assign 1 to x_2 and 0 to x_4... now solve for x_1, x_3

now solve for x_1, x_3

X says -2 times col 1 plus 1 times col 2 is the zero matrix!

* vector X is ~~a solution~~ in the null space, if is a solution to $Ux = 0$

We found 1 solution, $x = [-2 \ 1 \ 0 \ 0]'$,

What other vectors are in the null space? (ie are solutions to $Ax = 0$)

if $x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is in the nullspace, then so is any multiple... , $x = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Note X is a <u>line</u> in the null space

Now choose new values for free variables. Say $\underline{x_2 = 0}$, $\underline{x_4 = 1}$
(in general, for n free variables, set one at a time to 1 then zero all others)

$$X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \qquad \begin{aligned} x_1 + 2(0) + 2x_3 + 2(1) &= 0 \\ 2x_3 + 4(1) &= 0 \end{aligned}$$

$$x_3 = -2 \ , \quad x_1 = \cancel{4}\,2$$

set the free variables

this soln, $x = [2 \ 0 \ -2 \ 1]'$ says $2 \times col_1 + -2 \times col_3 + 1 \ col_4 = 0$ (from U)
so x is in the null space and is a solution to $Ux = 0$

Now we know what all solutions look like

$$\boxed{X = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}$$

So, the null space contains all combinations of "special" solutions. They form a plane! We chose to zero out all but 1 variable to get orthogonal vectors.

There are as many solutions as there are free variables.

Free Variables = # columns − # Pivots
(# of solns) (rank)

Recall, $U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in echelon form (upper triangular)

the Reduced Row Echelon form has 0's above and below pivots and pivots are normalized

$$U = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \boxed{\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R}$$

In matlab we can get reduced row echelon form using command rref(A)

The RREF clearly gives pivot rows and columns and contains identity matrix in pivot rows/cols

Now, look closer at R

$$I \quad \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \qquad \boxed{\begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}} \quad F$$

pivot cols   free cols

$0 \;\; 0 \qquad 0 \;\; 0$

RREF:

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \leftarrow r \text{ pivot cols}$$

r pivot columns   n-r pivot cols

to solve $Rx = 0$ for all null space ($RN = 0$):

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

$$Rx = 0 \rightarrow [I \;\; F]\begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \longrightarrow \boxed{x_{pivot} = -F\,x_{free}}$$

Example: this is the transpose of A from the 1st example

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

We expect to have 2 pivot cols and 1 free column, the 3rd col is linearly dependent on the first 2.

$\downarrow$ elimination

upper triangular

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\text{(bc no pivot)}]{\text{row exchange}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \boxed{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U}$$

pivot cols   free col

Rank = 2 (again)

\# pivot cols = 2 (again) $\longrightarrow$ \# pivots same for A, A^T

\# free cols = 1 $\longrightarrow$ \# free cols = \# cols − \# pivots

Now we solve for the null space vector x by setting the free variable to 1 and solving pivots

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$U \Rightarrow \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_2 + 2x_3 &= 0 \end{aligned}$$

$x_2 = -1$

$x_1 = -1$

$\uparrow x_3 = 1$

Nullspace: $x = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is a line

Now compute RREF

$$u = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{norm}} \boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R}$$

From $\underline{R}$ we see

$$R = \begin{matrix} \overbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}^{I} & \overbrace{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}}^{F} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{matrix}$$

From $\underline{x}$ we see

$$x = \begin{matrix} \boxed{\begin{matrix} -1 \\ -1 \end{matrix}} & -F \\ \boxed{1} & I \end{matrix}$$

So null space $N = c \begin{bmatrix} -F \\ I \end{bmatrix}$

HOW TO COMPUTE NULL SPACE

- Do elimination
  - pivot cols determines rank
  - free vars determine # of solns
- Continue elimination to RREF
- Nullspace, N, is $N = \begin{bmatrix} -F \\ I \end{bmatrix}$