COURSE REVIEW

Questions from old exams:

(1) Given $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has no soln

$Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has 1 soln

What do we know about m, n, r?

- we know # rows $= \underline{m = 3}$
- No solutions means $\underline{r < m}$
  (that is, more rows than pivots)
- 1 solution means nullspace only has zero vector
  that is $N(A) = \{0\}$ so $\underline{r = n}$
- Here's an example: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

(1-b) $\det(A^TA) = \det(AA^T)$? → NO A not square

(1-c) A^TA is invertible → YES $r=n$ full column rank (independent columns)

(1-d) AA^T is positive definite → NO AA^T is 3×3 but rank < 3

(1-e) Prove $\underline{A^Ty = c}$ has at least 1 soln for every c and has ∞ solutions LECT 8

- A^Ty has at least 1 soln because # rows of A^T (n) is equal to rank. Full row rank.
- $\text{Dim}\left[\text{Nullspace}(A^T)\right] = m-r$
  $m > r$, so 0 or ∞ solns

(2) $A = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$

(a) Solve $Ax = v_1 - v_2 + v_3$ for x

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(2-b) Suppose $v_1 - v_2 + v_3 = 0 = Ax$

then x is in nullspace of A

so solutions are <u>not</u> unique

(2-c) Suppose v_1, v_2, v_3 are orthonormal, what combination of v_1, v_2 is closest to v_3

orthonormal, look at projection $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

so, $\underline{0}\, v_1 + \underline{0}\, v_2 \approx v_3$

(3) Markov Matrix, find eigenvalues

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .2 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$$

$\leftarrow$ note: col 1 + col 2 = 2(col 3)

$\lambda_1 = 0$ bc singular

$\lambda_2 = 1$ bc Markov

$\lambda_3 = -.2$ bc trace $= .8$ so $\sum_i \lambda_i = .8$

(3-b) For $u_k = A^k u(0)$, $u(0) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ what does u_k approach?

$u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + c_3 \lambda_3^k x_3$ $\quad \lambda = 0, 1, -.2$

$u_\infty = c_2 x_2$

$u_\infty = c_2 \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$

let's find $x_2 \rightarrow \begin{bmatrix} -.8 & .4 & .3 \\ .4 & -.8 & .3 \\ .4 & .4 & -.6 \end{bmatrix} \begin{bmatrix} \uparrow \\ x_2 \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(can do elimination to solve for nullspace x_2)

$x_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \leftarrow$

(4) 2×2 matrix

(4-a) Find 1-D projection onto $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta}$$

(5) $\lambda_1 = 0$ $x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\lambda_2 = 3$ $x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ What is A?

$$A = S \Lambda S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$$

(6) Find A so that $A \neq B^TB$ for any B

B^TB is symmetric so find any non-symmetric matrix

(7) Find A w/ orthogonal eigenvectors but not symmetric

A could be skew symmetric $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

or orthogonal $\begin{bmatrix} c & -s \\ -s & c \end{bmatrix}$

(8) Least-Squares Solution to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 11/3 \\ -1 \end{bmatrix}$

(8-a) What is projection $\underline{p}$ of $\underline{b}$ onto columnspace of A?

$$\text{Proj} = \frac{11}{3} \text{Col } 1 + -1 \text{ col } 2$$

(8-b) Find different solution b so that $\begin{bmatrix} c \\ d \end{bmatrix} = 0$

let $b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ it is orthogonal to A