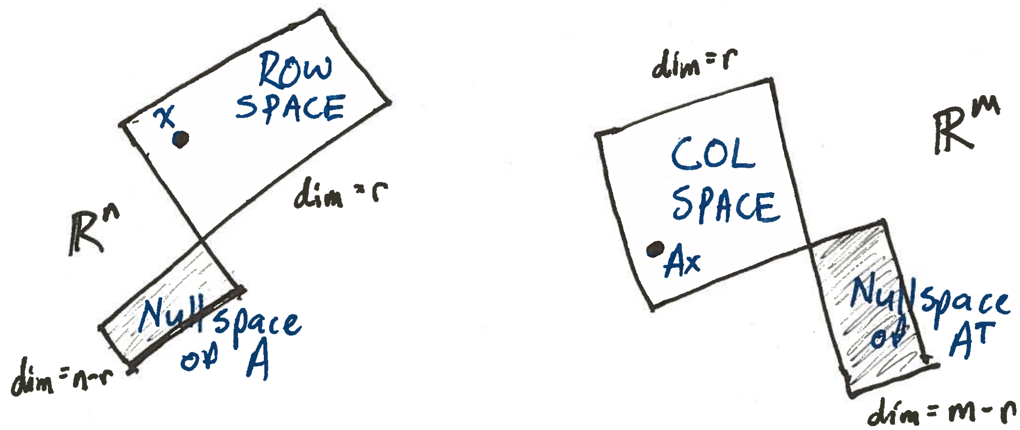
# 4 Subspaces, Left Inverse, Right Inverse, Pseudo Inverses




## INVERSE (2-SIDED)

$AA^{-1} = I = A^{-1}A$

* square matrix, full rank
$r = m = n$

## LEFT INVERSE

* If $\text{Rank}(A) = n$
then $\text{Rank}(A^TA) = n$
* 0 or 1 soln to $Ax=b$

* full column rank $r=n$
* Nullspace = $\{0\}$ bc cols independent

$$\underbrace{(A^TA)^{-1}A^T}_{\text{"left inverse"}} A = I$$

$$(A^{-1}_{left})\, A = I$$

## RIGHT INVERSE

* Infinite # solns to $Ax=b$
& $n-m$ free variables

* full row rank $r = m < n$
* Nullspace of $A^T = \{0\}$ because rows independent

$$A\,\underbrace{A^T(AA^T)^{-1}}\, = I$$

$$A(A^{-1}_{right}) = I$$

- If $x, y$ in rowspace then $Ax, Ay$ in columnspace and $Ax \neq Ay$ (1-to-1 mapping)

## PSEUDO INVERSE

- If $x \to Ax$ then $x = A^+(Ax)$ where $A^+$ is the pseudo-inverse

### Find Pseudo Inverse $A^+$

① Start from SVD: $\underline{A = U \Sigma V^T}$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \end{bmatrix}$$ $\Sigma$ is $m \times n$ w/ rank $r$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & & 0 \\ & \ddots & & & \\ & & 1/\sigma_r & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$$ $\Sigma^+$ is $n \times m$

$$\Sigma \Sigma^+ = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}$$ is $m \times m$ [Projection onto Colspace]

$$\Sigma^+ \Sigma = \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 0 \end{bmatrix}$$ is $n \times n$ [Projection onto rowspace]

* pseudo-inverse does not produce identity matrix (ideal) but projects into row/column space (next best thing)

$$\underline{A^+ = V \Sigma^+ U^T}$$