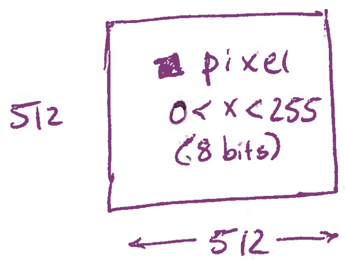
CHANGE OF BASIS, COMPRESSION OF IMAGES

IMAGE COMPRESSION



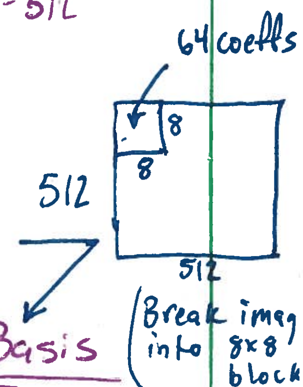
$X \in \mathbb{R}^n$

$n = 512^2$

(grayscale)

$$X = \begin{bmatrix} \vdots \\ 121 \\ 124 \\ \vdots \end{bmatrix} \qquad \|x\| = 512$$

JPEG uses Fourier Basis



Standard Basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Better Basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$$

Fourier Basis

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^7 \end{bmatrix},$$

see lect 24

JPEG

$$\text{SIGNAL} \xrightarrow[\text{lossless}]{\text{change basis}} \text{COEFFS} \xrightarrow[\text{lossy}]{\text{compression}} \text{COMPRESSED}$$

Wavelet Basis $\mathbb{R}^8$

(all vectors are orthogonal!) → So W^{-1} is fast!

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{w_1}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{w_7}, \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{w_8}$$

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_8 \end{bmatrix} = c_1 w_1 + \dots + c_8 w_8 \quad \text{(change of basis)}$$

$$= \begin{bmatrix} W \end{bmatrix} \begin{bmatrix} \uparrow \\ c \\ \downarrow \end{bmatrix} \qquad (p = Wc \rightarrow c = W^{-1}p)$$

CHANGE OF BASIS

Must Be : ① Fast FFT (Fast Fourier Transform)
FWT (Fast Wavelet Transform)

② Must be able to throw out a few basis vectors b.c., a few accurately describe signal (compression)

Idea: Let columns of W = new basis vectors

$$\begin{bmatrix} x \end{bmatrix}_{\text{old basis}} \longrightarrow \begin{bmatrix} c \end{bmatrix}_{\text{new basis}} \qquad \underline{x = Wc}$$

Given a linear transformation T ($T: \mathbb{R}^8 \to \mathbb{R}^8$)

w/ respect to a basis $v_1, \ldots, v_8$ it has a matrix A

w/ respect to a basis $w_1, \ldots, w_8$ it has a matrix B

We will compute a transformation in these 2 bases, but there must be a connection between A and B. They are similar!

SIMILAR: $B = M^{-1} A M$

where here, M is the change of basis matrix.

Notice, When ~~which~~ we change basis (v, w), every vector (x) has new coordinates (c)

· In these types of problems, a transformation matrix is given (T) and the basis vectors (x), we want to solve for matrix A so that

$$T = Ax$$

And A gives us a change of basis from T to x

What is A? using basis $v_1, \dots, v_8$

- We know T completely if we know how T behaves on the basis vectors. That is, we can solve for T in all space by knowing only 8 values, $T(v_1), \dots, T(v_8)$. We can do this because of linearity! Every vector is a linear combination of the basis vectors

- Because every

$$X = c_1 v_1 + c_2 v_2 + \dots + c_8 v_8$$

We know

$$T(X) = c_1 T(v_1) + \dots + c_8 T(v_8)$$

- So let's write

$$T(v_1) = a_{11} v_1 + a_{21} v_2 + \dots + a_{81} v_8$$

$$T(v_2) = a_{12} v_1 + a_{22} v_2 + \dots + a_{82} v_8$$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & \\ \vdots & \vdots & \cdots \\ a_{81} & 8_{82} & \end{bmatrix}$$

so $T = Av$

transformation vectors

basis vectors