POSITIVE DEFINITE MATRIX (TESTS)
TESTS FOR MINIMA ($x^TAx > 0$), ELLIPSOIDS

GIVEN A SYMMETRIC MATRIX WE CAN TEST IF POS. DEF.

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

1. EIGENVALUE TEST
   $\lambda_1 > 0, \quad \lambda_2 > 0$
2. DETERMINANT TEST
   $a > 0, \quad ad - b^2 > 0$
3. PIVOT TEST
   $a > 0, \quad \frac{ad-b^2}{a} > 0$
4. REAL POS DEF TEST (SYMMETRY)
   $x^TAx > 0$

EXAMPLE

we will test out a couple of matrices by swapping the a_{22} entry

1. $$A_1 = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$$

Note: this matrix is singular! (col 1 is multiple of col 2) so rank 1, 1 pivot, AND WE KNOW $\lambda_1 = 0$

SO By trace $= \sum_i \lambda_i$, we know $\lambda_2 = 20$

SINCE $\lambda_1 = 0$ THIS MATRIX IS NOT POS DEF BUT IS POSITIVE SEMI-DEFINITE

$A_1 = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ Let's use our formal test

$$x^T A x = [x_1 \; x_2] \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \; x_2] \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$$

$$= \underbrace{2x_1^2}_{a x^2} + \underbrace{12 x_1 x_2}_{2b\, xy} + \underbrace{18 x^2}_{c y^2} \quad \text{(quadratic)}$$

IN FORM → $a x^2$ $\quad 2b\, xy \quad$ $c y^2$

Now let's let entry $a_{22} = 20$ so we are positive definite

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \rightarrow x^T A x = 2x_1^2 + 12 x_1 x_2 + 20 x_2^2$$

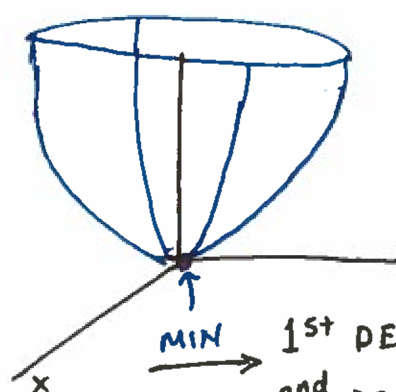
THIS MATRIX IS NO LONGER SINGULAR SO $\lambda \neq 0$

WE KNOW $\lambda > 0$ BC $\underbrace{TR(A) = 22 > 0}_{= \lambda_1 + \lambda_2}$ AND $\underbrace{DET(A) = 4 > 0}_{= \lambda_1 \cdot \lambda_2}$

SO $\lambda_1, \lambda_2 > 0$ AND MATRIX IS POS. DEF.

THIS FUNCT IS POS EVERYWHERE EXCEPT FOR $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$F(x,y) = 2x^2 + 12xy + 20y^2$$



→ 1^{st} DERIV ZERO

2^{nd} DERIV > 0

MINIMUM TEST

- CALC WE SAY $\dfrac{d^2 u}{dx^2} > 0$

- LA WE SAY MATRIX IS POSITIVE DEFINITE

WITHOUT LOOKING AT A GRAPH OF OUR FUNCTION WE CAN STILL SEE IF ALWAYS POSITIVE BY EXPRESSING AS SUM OF SQUARES (THEN NEVER NEG.)

$$P(x,y) = 2x^2 + 12xy + 20y^2$$
$$= 2\left(x^2 + \underline{3}\,y\right)^2 + 2y^2$$

NOTICE THIS WILL NEVER BE NEGATIVE.

THESE NUMBERS DO NOT HAPPEN BY ACCIDENT...

IF WE DO ELIMINATOON ON A:

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} = U$$

$$A = LU \longrightarrow \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$$

* COMPLETING THE SQUARE IS ELIMINATION!

THIS IS NICE BC COMPLETING SQUARES IS OK FOR SMALL SYSTEMS BUT ELIMINATION IS MUCH BETTER WHEN USING LARGE MATRICES.

THIS CONFIRMS THE FACT THAT POS. PIVOTS GIVES POS. DEF.

Second Derivative in Matrix notation:

$$A'' = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

* this matrix is symmetric BC $f_{xy} = f_{yx}$.

3x3 EXAMPLE

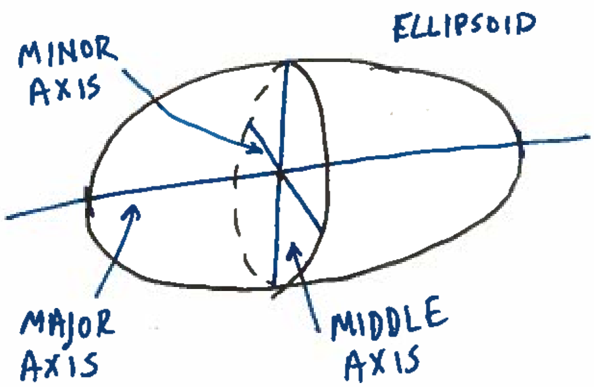
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

IS A POSITIVE DEFINITE ?

1. DETERMINANTS ALL POSITIVE
2. PIVOTS ALL POSITIVE $(2, 3/2, 4/3)$
3. EIGENVALS POSITIVE $(2, 2 \pm \sqrt{2})$
4. $x^T A x$

$$= 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$$

LET'S TAKE FUNCTION FROM 4
AND SOLVE IT AT SOME CROSS-SECTION, SAY $\Rightarrow F = 1$
WE HAVE AN EQUATION OF AN ELLIPSOID.



THE AXES :

- ARE IN DIRECTION OF EIGVECTORS
- HAVE LENGTHS OF EIGVALUES

PRINCIPAL AXIS THM:

$$A = Q \Lambda Q^T$$