COMPLEX VECTORS AND MATRICES, INNER PRODUCTS FOURIER MATRIX, FOURIER TRANSFORM (DFFT)

RULES IN CMPLX SPACE

FOR COMPLEX MATRICES WE CALCULATE LENGTH AS

$$\boxed{|z| = \bar{z}^T z = z^H z}$$

(bar is conjugated)
(H is Hermitian, conjugate + Transpose)

INNER PRODUCT: (A,B complex)

$$A \cdot B = \bar{B}^T A = B^H A$$

SYMMETRIC MATRICES (complex)

SYMMETRIC IF $\bar{A}^T = A = A^H$
(HERMETIAN)

EXAMPLE:

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Note, diagonal entries must be real!

This is a Hermitian Matrix

PERPENDICULAR

PERPENDICULAR IF FOR Vectors

$q_1, q_2, \ldots, q_z$

WE HAVE

$$\bar{q}_i^T q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

EXAMPLE:

$$Q = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ q_1 & q_2 & \cdots & q_z \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\bar{Q}^T Q = I = Q^H Q$$

UNITARY (C^n) <=> ORTHOGONAL (R^n)

FOURIER MATRIX

*Note, we use EE notation for the rest of this lecture so that column 1 is now column 0, so we have n rows/cols numbered from 0 to n-1

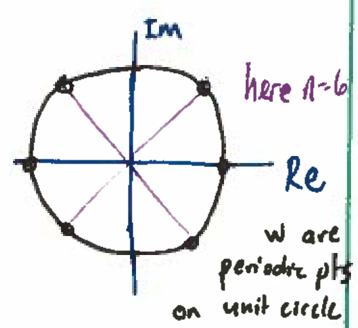
$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & & W^{n-1} \\ 1 & W^2 & W^4 & & W^{2(n-1)} \\ \vdots & & & & \\ 1 & W^{n-1} & W^{2(n-1)} & \cdots & W^{(n-1)^2} \end{bmatrix} \Rightarrow (F_n)_{ij} = W^{ij}, \quad i,j = 0 \text{ to } n-1$$

← this matrix is symmetric

WHAT IS THE NUMBER W?

WE WANT $W^n = 1$

WHICH IS SATISFIED BY $W_n = e^{i 2\pi/n}$



4×4 FOURIER MATRIX

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

IMPORTANT: THE COLS OF THIS MATRIX ARE ORTHOGONAL

TO SEE THIS TAKE INNER PRODUCT OF ANY (AND ALL) 2 COLS USING FACT $A_i \cdot A_j = \bar{A}_j^T A_i$, $\perp$ IF ZERO

THIS FACT MEANS ITS EASY TO INVERT F

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} P$$

WHERE

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & & & \\ 1 & 0 & 1 & & & \\ 0 & 1 & 0 & 1 & & \\ \vdots & & & & & \end{bmatrix}$$

PERM MATRIX

$$D = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{31} \end{bmatrix}$$

\# CALCULATIONS $= \cancel{64^2}$

$$= 2(32)^2 + 32$$

IF WE DECOMPOSE 32 MATRIX TO 16

\# CALCULATIONS

$$= 2\left(2(16)^2 + 16\right) + 32 \xrightarrow{\text{CONTINUING}} \underline{\tfrac{1}{2} n \log_2 n}$$

CONSIDER CASE $N = 1024$

$$n^2 = 2^{20} > 1 \text{ million}$$

$$\frac{n}{2}\log_2 n = 5 \cdot 1024 \sim 5{,}000$$ MUCH BETTER!