SYMMETRIC MATRICES, EIGENVALUES + EIGENVECTORS

START: POSITIVE DEFINITE MATRICES

SYMMETRIC MATRICES (most important class of matrix)

A matrix is symmetric when $A = A^T$

1. THE EIGENVALUES ARE REAL
2. THE EIGENVECTORS ARE PERPENDICULAR

WE CAN WRITE ANY MATRIX AS: $A = S \Lambda S^{-1}$

FOR SYMMETRIC MATRICES: $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$

WHERE S IS A MATRIX OF EIGENVECTORS

AND Λ IS A DIAGONAL EIGENVALUE MATRIX

AND Q IS ORTHONORMAL EIGENVALUE MATRIX

WHY ARE THE EIGENVALUES (λ's) REAL?

$$Ax = \lambda x \xrightarrow[\text{can always do}]{\text{conjugate}} \bar{A}\bar{x} = \bar{\lambda}\bar{x} \xrightarrow{\text{transpose}}$$

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda} \xrightarrow{\text{Symmetry}} \bar{x}^T A = \bar{x}^T \bar{\lambda} \xrightarrow{\text{Multiply by}}$$

$$\bar{x}^T A x = \bar{x}^T \bar{\lambda} x$$

$\longrightarrow \lambda = \bar{\lambda}$ so λ must be real (imag part is 0)

$$\boxed{\text{SYMMETRIC IS } A = A^T \text{ SO } A = Q\Lambda Q^T}$$

THAT IS,

$$A = Q\Lambda Q^T = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ q_1 & q_2 & \cdots & q_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \leftarrow & q_1 & \rightarrow \\ \leftarrow & q_2 & \rightarrow \\ \leftarrow & q_n & \rightarrow \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \dots + \lambda_n \underbrace{q_n q_n^T}_{\text{a projection matrix}}$$

EVERY SYMMETRIC MATRIX IS A COMBINATION OF PERPENDICULAR PROJECTION MATRICES

FACT

FOR SYMMETRIC MATRICES: the signs of the pivots are the same signs as the eigenvalues.

↳ # positive pivots = # positive λ's

POSITIVE DEFINITE MATRICES

- A positive definite matrix is symmetric w/ positive eigenvalues (and all the pivots are positive)
- All subdeterminants are positive

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$