EIGENVALUES, EIGENVECTORS, $\det[A-\lambda I]=0$, TRACE $=\lambda_1+\dots+\lambda_n$

A matrix A acts on a vector, x. We input a vector x and out comes a vector Ax, it's like a function. We are interested in the vectors that come out parallel to Ax, these are called the eigenvectors.

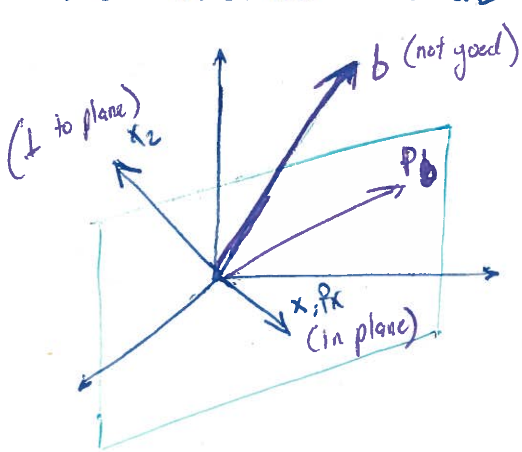
$$Ax = \lambda x$$ [this says the eigen vectors are parallel (in same direction) to Ax]

Here $\underline{x}$ is the $\underline{\text{eigenvector}}$, $\underline{\lambda}$ is the $\underline{\text{eigenvalue}}$

EIGENVALUE + NULLSPACE

We've already seen this for eigenvalues $\underline{\lambda=0}$. These eigenvectors x were in the $\underline{\text{nullspace}}$ of Ax (we solved $Ax=0$).
if A is singular, $\lambda=0$ is the eigenvalue.

EIGENVECTORS + EIGENVALUES OF PROJ MATRIX



What are the x's and λ's for proj. matrix?

- For any x in plane:

$$Px = x \quad , \quad \text{so } \lambda = 1$$

- For any $x \perp$ plane

$$Px = 0 \quad , \quad \text{so } \lambda = 0$$

EXAMPLE

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the eigenvectors (ie a vector that we can multiply by and end up in the same direction)?

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and has $\lambda = 1$, $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $Ax = x$

$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and has $\lambda = -1$, $Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $Ax = -x$

we determined the eigenvectors by inspection but will develop systematic ways to calculate them in the future.

TIP:
- $n \times n$ matrix will have n eigenvalues
- sum of eigenvalues will equal sum down diagonal of A (trace)

FACT: $\sum_n \lambda_n = \sum_n a_{nn}$ (sum of eigenvals is trace of A)

HOW TO SOLVE $Ax = \lambda x$?

- Re-Write as $(A - \lambda I)x = 0$

→ we don't know λ or x but IF we can solve it, $\underline{A - \lambda I}$ matrix must be singular. The determinant of singular matrices is always zero.

$$\det(A - \lambda I) = 0$$

So now we have an eqn for λ that does not involve x!

→ When we find the eigenvalues (λ) we can substitute back into $(A - \lambda I)x = 0$ and solve for the nullspace which gives us the eigenvectors, x!

EXAMPLE SYMMETRIC MATRIX

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ Find Eigenvalues and Eigenvectors

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 2) = 0 \qquad \lambda_1 = 4,\ \lambda_2 = 2$$

* Note! this is $\lambda^2 -$ TRACE(A) + DET(A)

$(A - \lambda I)x = 0$

(1) $\lambda_1 = 4$ $\quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(2) $\lambda_2 = 2$ $\quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Comparing the eigenvals/eigenvects from our previous example, we see the x's stayed the same but the λ's increased by 3. Formally we see,

if $Ax = \lambda x$

then $(A + 3I)x = \lambda x + 3x = (\lambda + 3)x$

This is a very cool result!

FACT: $\prod_n \lambda_n = \text{DET}(A)$ (product of eigenvals is determinant of A)

EXAMPLE ROTATION MATRIX (90° rotation)

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

WE WILL HAVE PROBLEMS BC NO REAL VECTOR CAN ROTATE AND BE || TO ITSELF.

EIGENVALS: $\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$ $\quad \underline{\lambda_1 = i}, \underline{\lambda_2 = -i}$

This matrix is anti-symmetric and difficult to work with

EXAMPLE TRIANGULAR MATRIX

The eigenvals of a triangular matrix are easy! They are on diagonal

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

WE WILL HAVE PROBLEMS FINDING A UNIQUE EIGENVECTORS BC REPEATED ROOTS.

EIGENVALS: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$ $\quad \lambda_1 = 3, \lambda_2 = 3$

EIGENVECTS: $(A - \lambda_1 I)x_1 = 0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$(A - \lambda_2 I)x_2 = 0 \Rightarrow \lambda_1 = \lambda_2$ so x_2 is <u>not</u> unique