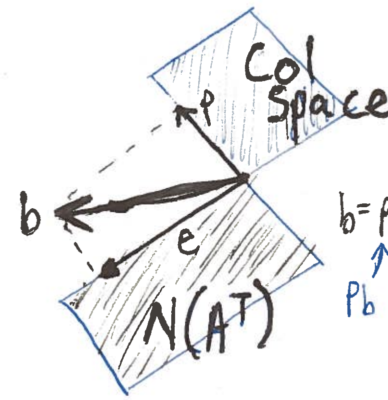
Projection Matrix, Least Squared continued

Projection Matrix: $P = A(A^TA)^{-1}A^T$

(1) If b in column space, $Pb = b$
(2) If $b \perp$ to column space, $Pb = 0$



$b = p + e$

$\uparrow \quad \uparrow$

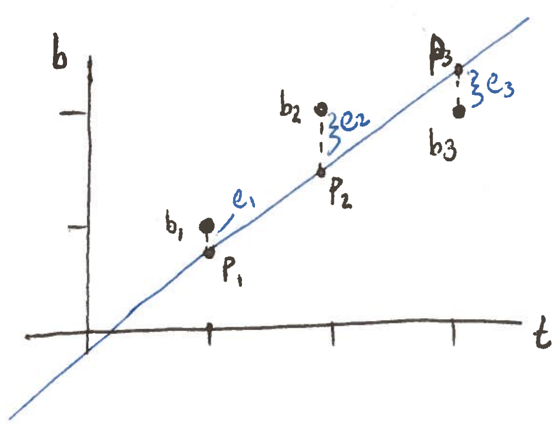
$Pb + (I-P)b$

Least Squares (from Lec 15)

$Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

- No solution
- We have basis for colspace BUT does not include b!
- Minimize error $\|Ax-b\|^2 = \|e\|^2$



- P's are pts on line, they are in the column space!
- b's are original points

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P we have $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

*start here $\underbrace{A^TA}\hat{x} = \underbrace{A^Tb}$

A^TA: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

A^Tb: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$$3\hat{C} + 6\hat{D} = 5$$
$$6\hat{C} + 14\hat{D} = 11$$

[normal equations]

Continue solving:

$$\left[\begin{array}{cc:c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 0 & 2 \\ 0 & 1 & 1/2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 1/2 \end{array}\right]$$

So, $\underline{D = 1/2}$, $\underline{C = 2/3}$

which means $\boxed{b_p = \frac{2}{3} + \frac{1}{2} t}$ is my best fit!

t	p	b	e
0	2/3	–	
1	7/6	1	-1/6
2	5/3	2	2/6
3	13/6	2	-1/6

$$b = p + e$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} + \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

$$\boxed{\begin{array}{c} A^T A \hat{x} = A^T b \\ p = A\hat{x} \end{array}}$$ Key Eqns

- p and e are perpendicular!
- e is in col space of A

<u>FACT:</u> If A has independent cols then $A^T A$ is invertible

<u>Proof:</u> Suppose $A^T A x = 0$, then x must be 0

(a matrix is invertible when its nullspace only contains zero vector)

let's take dot product w/ x

$$x^T A^T A x = 0$$

$$(Ax)^T A x = 0 \quad \text{so} \quad Ax = 0$$

and since cols of A independent, then only thing in nullspace is zero vector so x must be 0