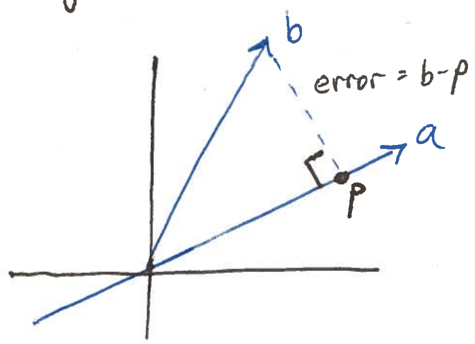
Projection, Least Squares, Projection Matrix

Projection:



projection, p, is some multiple of a,

$$p = xa$$

and $a \perp$ error, [a · error = 0]

$$a^T(b - xa) = 0$$

solve for x, projection mult of a,

$$\boxed{x = \frac{a^T b}{a^T a}} \rightarrow \boxed{p = a\frac{a^T b}{a^T a}}$$

In calculus the projection has trig function soln's in linear algebra the projection is a matrix: $p = a\frac{a^T b}{a^T a} = \mathbf{P}b$

Projection matrix, P,

$$\boxed{\mathbf{P} = \frac{a a^T}{a^T a}}$$

so projection p is $\underline{p = Pb}$

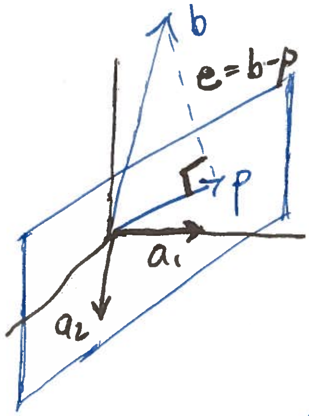
Notes about projection matrix, P:

- Column space $C(P)$ is a line through a
- rank $(P) = 1$, it's a line
- P is symmetric, $\underline{P^T = P}$
- Projecting more than once produces no change, $\underline{P^2 = P}$

Why use projection?

if $Ax = b$ has no solutions (say, more unknowns than eq'ns)
then we solve $A\hat{x} = p$ (p is projection of b into colspace of A)

Now consider 3-dimension case:



- project vector b into plane
- Describe plane using 2 basis vectors

$$\text{plane of } a_1, a_2 = \text{colspace of } A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

- error ($e = b - p$) is perpendicular to p
- projection, p, is some multiple of plane

$$p = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$$

Projection:

$p = A\hat{x}$, we want $\hat{x}$

Key is: error $= e = b - p = \underline{b - A\hat{x}}$ (error $\perp$ plane) [$a \cdot$ error $= 0$]

$$a_1^T(b - A\hat{x}) = 0$$

$$a_2^T(b - A\hat{x}) = 0$$

$$\downarrow$$

$$A^T(\underbrace{b - A\hat{x}}_{e}) = \emptyset$$

e is in nullspace of A^T (from $A^Te = 0$)

$\hookrightarrow e \perp C(A)$!

$$\downarrow$$

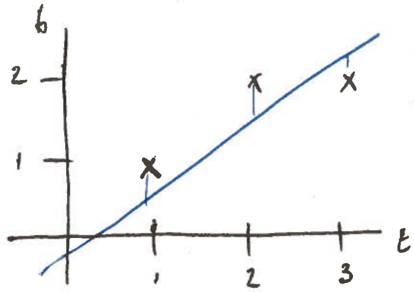
$$\boxed{\hat{x} = (A^TA)^{-1}A^Tb} \longrightarrow p = A\hat{x} = A(A^TA)^{-1}A^Tb$$

$$\downarrow$$

$$\boxed{P = A(A^TA)^{-1}A^T}$$

so $\boxed{p = Pb}$

Least Squares (fitting by a line)



$b = C + Dt$

$C + D = 1$

$C + 2D = 2$

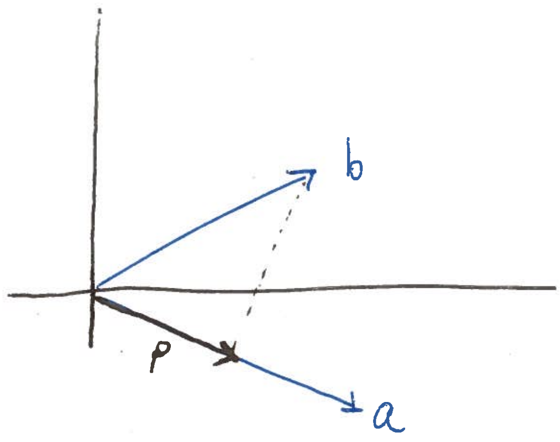
$C + 3D = 2$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

So, we can't solve $Ax = b$, but we can solve the next best thing, $Ax = p$ . . . $A^T A x = A^T b$

The DOT PRODUCT in Matrix notation

$$\boxed{a \cdot b = B^T A}$$



given 2 vectors, **a** & **b**
the projection of a onto b
is given by the dot product

$$a \cdot b = |a||b| \cos\theta$$

Say $\Rightarrow$ $a = (1,2,4)$ $b = (-2,4,-1)$

then $a \cdot b = 1 \cdot -2 + 2 \cdot 4 + 4 \cdot -1$

$= -2 + 8 - 4$

$= 2$

In matrix form,

$$A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$$

$$B^T A = \begin{bmatrix} -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$= -2 + 8 - 4$

$= 2$