Bases of vector spaces, rank 1 matrices

Bases of new vector Spaces:

Say we have a vector space M = all 3×3 matrices $\dim(M) = 9$

A subspace might be: S = symmetric 3×3 $\dim(S) = 6$

U = upper triangular 3×3 $\dim(U) = 6$

A basis for M might include

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ldots \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

consider new space . . . . $\boxed{\text{INTERSECTION}}$

$D = S \cap U$ = symmetric and upper triangular

= diagonal matrices

$$\dim(S \cap U) = 3$$

recall, we didn't care about union $S \cup U$ because it is not a subspace, just 2 lines in plane

Another new space . . . $\boxed{\text{SUM}}$

$A = S + U$ = sum of any element of S, U

= all 3×3 matrices!

$$\dim(S + U) = 9$$

<u>FACT:</u> for any 2 subspaces, S, U

$$\dim(S) + \dim(U) = \dim(S \cap U) + \dim(S + U)$$

from our example we see: $6 + 6 = 3 + 9$ ✓

<u>Example</u>:

Given diff eqn $\boxed{\frac{d^2y}{dx^2} + y = 0}$,

solutions look like $y = \cos(x), \sin(x), e^{ix}$

the complete solution is $\boxed{y = c_1 \cos(x) + c_2 \sin(x)}$

which is a <u>vector space</u>. A <u>basis</u> for this vector space is $\boxed{\cos x, \sin x}$. $\dim\left(\substack{\text{soln} \\ \text{space}}\right) = 2$

Consider a rank 1 matrix

$$\underset{2\times 3}{A} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

$d(C(A)) = 1$ so rank 1

basis $(A) = (1, 4, 5)$

ALL RANK 1 MATRICES LOOK LIKE: $A = uv^T$

I can create a rank N matrix from N rank 1 matrices

Example:

Say, in R^4 $\quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$, $S =$ all V in R^4 w/ $v_1 + v_2 + v_3 + v_4 = 0$

Is S a subspace? YES! $\quad c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$

What's the dimension of S? $\quad \dim(S) = 3$

What's special about S? $\quad S$ is the ~~null~~space of $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T$
where $AV = 0$ (nullspace)

Rank$(A) = 1$, $\dim(N(A)) = n - r = 4 - 1 = 3$

A basis of the null space, S, is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$ use free var's

The column space of A is $R^1 = C(A)$

The $N(A^T)$ is $\{0\}$

Check Dimensions!

null space, row space

$$\dim(N(A)) + \dim(C(A^T)) = 3 + 1 = 4 = n \text{ (\#cols)}$$

$$\dim(C(A)) + \dim(N(A^T)) = 1 + 0 = 1 = m \text{ (\#rows)}$$

column space, left space